

Statement: Let $K \in \mathbb{Z}_+$. Let a binary string be generated by first letting $s_1 = s_2 = \dots = s_K = 0$. We then let $s_i = 1$ for $i > K$ iff there exists some non-empty string t such that the last $|t| \cdot K$ characters of $s_1, \dots, s_{i-1}$ equal $t + t + \dots + t$ where $|t|$ denotes the length of t and $+$ denotes concatenation.

Theorem: Define $T_{K,N} := (\dots((0^K 1)^K 1)^K \dots)^K 1$ followed by all ones where c^K denotes the character repeated K times and $\dots$ means there are N ones in total in the expression. Then the resulting string will be $T_{K,K}$. For $K = 1$ this is 01 followed by all ones, for $K = 2$ it is $(0^2 1)^2 1 = (001)^2 1 = 0010011$ followed by ones and so on.

Proof: $K = 1$ is trivial so we assume $K \geq 2$. We do this by induction, so we only have to prove that if s is equal to the proposed string up to character i , they will also agree on character $i + 1$. The base case is trivial as both strings start with 0^K by definition. Furthermore once $i > |T_{K,K}|$ we must get an infinite sequence of ones as $T_{K,K}$ ends in 1^K . Thus we only need to consider $K \leq i < |T_{K,K}|$. We note by the definition of K that any 1 is clearly preceded by a repeat of K equal strings, so we only have to show the converse. Say we have some such substring t that repeats K times. We will show that the character following t is 1. Let j be the highest number of consecutive 1s in t^K . Any substring in $T_{K,K}$ not containing $j + 1$ consecutive 1s must lay between two instances of 1^{j+1} found within $T_{K,j+1}$. Now t^K must contain at least $K - 1$ instances of at least j consecutive 1s, but laying between to 1^{j+1} it can touch at most $K + 1$ such instances. We have a few cases, suppose first we have $K - 1$ instances of at least j consecutive 1s in t^K . Then t does not contain 1^j so t must start with 1^a and end with 1^b such that $a + b = j$. But this means the first 1^a and the last 1^b touch the 1^{j+1} instances. Since $b < j + 1$ this means the character following t^K is 1, so we are done with this case. Thus we can assume there are K instances of 1^j in t^K . Since there are most $K + 1$ total that t^K can touch with two being at either end, one of the instances has to be a prefix or suffix of t^K since it can't miss out on both of them. Thus t^K ends or starts with 1^j , so t does as well since it's not all 1s. Consider first the case when t starts with 1^j . Since t^K does not contain 1^{j+1} we see that t must end on a zero. Any two ones in $T_{K,K}$ are separated by 0^K so since t^K appears in $T_{K,K}$ this means t ends with 0^K . But then t^K ends with 0^K , so the next character after t^K is a 1, which completes this case. Thus we can consider the other case when t ends with 1^j . But then the last instance of 1^j in t^K touches the instance of 1^{j+1} , so the next character is a 1 as well in this case. Thus all cases are covered, completing our proof.