2023 Rocky Mountain Regional Solutions

The Judges

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Credits

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Attendance First solve: 3m

Problem

Roll call! Determine which students do not respond to their name being called.

Solution

Pretty straightforward, one approach is to check every time a name is called if the previous call was a name or `Present!` to see if the previous name was present or not.

Be careful, when the input ends have you handled the last callout carefully (i.e. was it a name?)

Guess Who — First solve: 9m

Problem

Given a list of characters defined by M Boolean attributes, and a list of queries of the attributes, identify the character(s) that are consistent with the queries.

Solution

- It is sufficient to check each character against each query, and note who is consistent with the queries.
- Since all queries are on unique attributes, we can combine the queries and check them in $O(M)$ time for each character.
- $O(NM)$ overall.

Don't Be Fake — First solve: 9m

Problem

Given N lists of intervals, find the times of maximum overlap.

Solution

- Since the problem is small, we can simply scan every possible second and check how many intervals contain that time.

Bombardment First solve: 21m

Problem

Simulate a greedy strategy for bombarding a set of points on a line.

Solution

Can compute the best bombardment in $O(n)$ time (after sorting the points). Keep track of the left- and right-most points in the current "window" as you scan through the points.

The problem statement guarantees the window is really wide. Can show each iteration will remove a large fraction of the points, so the number of iterations is very small.

Thwack! First solve: 1h23m

Problem

Find all winning opening moves in a 1D game of stones. A stone can capture an adjacent stone of another color.

Solution

Classic Sprague-Grundy numbering. For a connected region of stones, any move splits it into at most 2 regions.

Compute *nimbers* for a connected region C as follows.

- $f(C) = 0$ if there are no valid plays (or C is empty).
- Otherwise, try all possible plays and let C', C'' be the two regions that result.
- Let $f(C)$ be the minimum nonnegative integer that cannot be obtained of the form $f(C')$ XOR $f(C'')$ over all valid plays.

Thwack! (cont.)

Solution

- A game board is a first-player loss if and only if the `XOR` if all $f(C)$ values for all connected regions is 0.
- Only $O(n^2)$ possible connected regions can be seen since each is a substring of the input apart from, perhaps, two endpoints. Using memoization, this leads to an $O(n^3)$ algorithm but even $O(n^4)$ would pass.

Gas Station — First solve: 1h33m

Problem

Given the number of gas pump at a gas station, information about all the cars coming to refuel, and rules about the behaviour of the cars, simulate the cars refuelling at the gas station.

Solution

Teams need to be careful and understand the rules when implementing their solution. It is important that the order of operations is correct when cars are leaving at the same time a car is arriving.
At any time X, first all cars that are done refuelling leave, then any cars in queue move into open lanes, then finally an arriving car looks at queue lengths to figure out where to queue up.
The bounds on this problem are small, so that a correct solution should pass, without needing to optimize the run time.

Problem

Given the rules of a card solitaire game:

- what happens in a turn depends on what card is drawn next
- determine the expected number of turns to end the game, out of the 45! possible orderings.

Solution

- If we were to write a recursive function to simulate one move, we would try to draw each possible card, recurse, and weigh the result by the probability of drawing that card.
- We would need to know:
  - up/down state of the 7 cards (2^7)
  - how many of 1s, 2s, . . . , 7s, and the rest are left ($5^7 \times 24$)

Seven Up (cont.)

Solution

- This is relatively small, so we can use dynamic programming to remember values already computed.
- An optimization (optional): if card at position k is face up, then all card k can be considered the "rest". Reduce the number of states significantly.

A Complex Problem — First solve: 2h52m

Problem

Given a directed graph with two types of edges representing subset and strict subset relations, find the minimum and maximum number of distinct sets among those specified.

Solution

- If there is any cycle of subsets, all sets in the cycle are equal. Otherwise, any two sets with no cycle connecting them can be distinct.
- To find the maximum number of distinct sets, count the number of strongly connected components in the graph, treating both edge types the same way.

A Complex Problem (cont.)

Solution

- Once we [conceptually] collapse strongly connected components into single nodes, we are left with a directed acyclic graph.
- If there is a path with a strict subset relation connecting two nodes, then the two nodes are distinct.
- To find the minimum number of distinct sets, find the longest weighted path in the directed acyclic graph by treating subset edges as having weight 0 and strict subset edges as having weight 1, and add one.
  - We can solve this with dynamic programming by finding the longest weighted path ending at each node.
  - For each node, recursively find the maximum among zero, the maximum of its subset children, and one plus the maximum of its strict subset children.

Ribbon Road — First solve: 3h9m

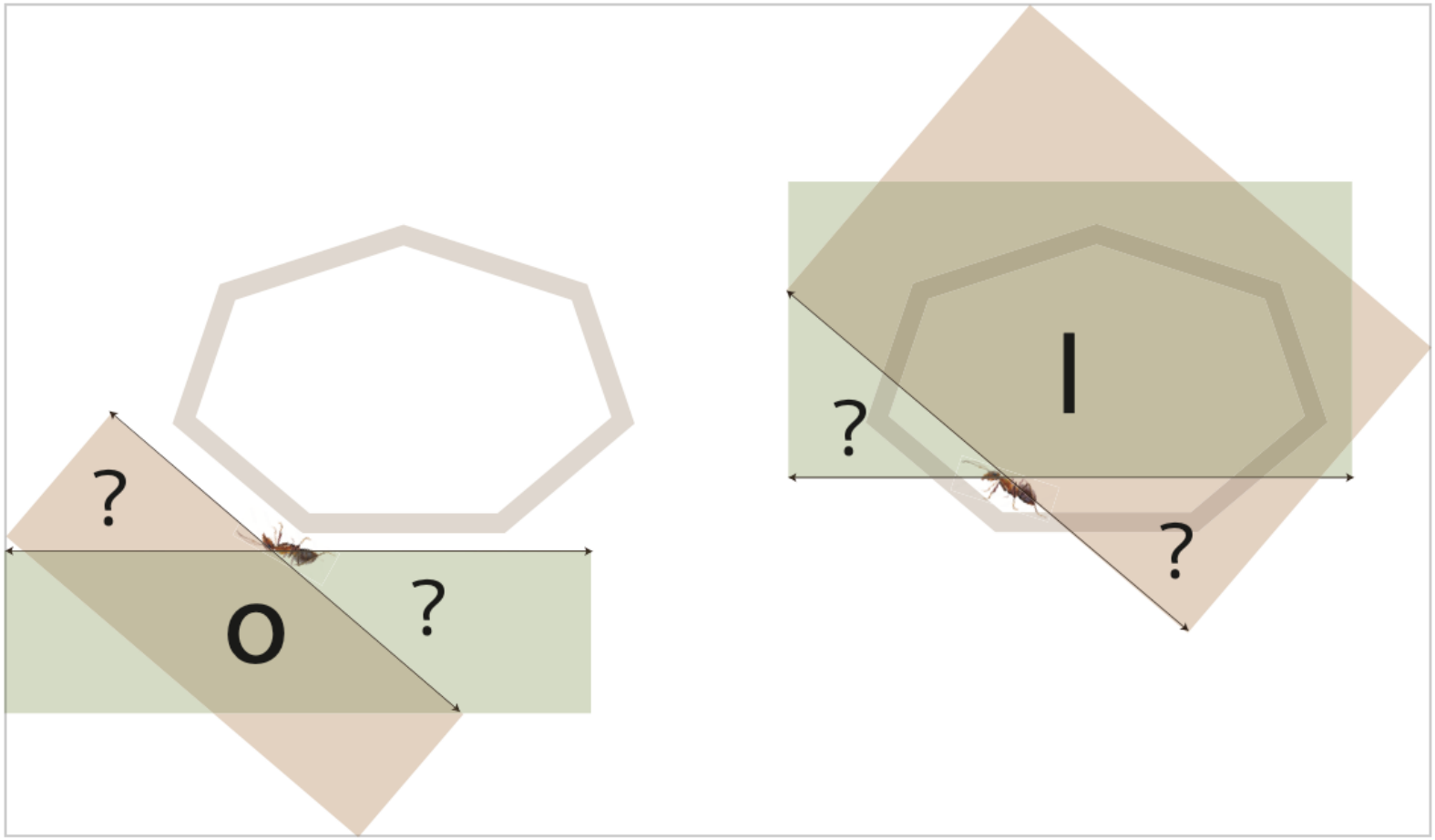
Problem

Given a point on a polygon and a ray from that point, is the ray pointing to the "inside" or "outside"?

Solution

- "inside" and "outside" is defined with respect to the edge containing the point
- determine if the orientation of the polygon is consistent with the orientation of the direction of the ray
- If ray is parallel to edge, then it is ambiguous.
- at a corner, test both edges. The result may be inside, outside, or ambiguous.

Ribbon Road (cont.)



Add or Multiply

Problem

Given an initial mathematical expression consisting of N digits, $+$'s, and $\times$'s and M queries: *swap*, *flip*, and *all-flip*, evaluate the initial expression and re-evaluate the expression after each query.

Solution

- Observations:
  - Naive approach: $O(NM)$ — too slow
  - For *swap* and *flip*, we need to identify the terms (consecutive $\times$'s) around the blocks she changes to update the total value efficiently.
  - For *all-flip*, we need a different strategy.

Add or Multiply (cont.)

Solution

- Maintain the total values of two phases — preparing for *all-flip*
- Use segment trees for efficient *swap* and *flip*.
  - Approach 1:
    - One segment tree for multiplication.
    - Other segment trees to keep track of term boundaries (e.g. indices of the right-most $+$, etc.)
    - For each query, we subtract the value of old terms and add the value of new terms to the total value.
  - Approach 2:
    - View a fragment of an expression as either one term $(\times \ldots \times z)$ or more than one term $(\times \ldots \times a + (b \ldots) + (c \times \ldots \times z))$.
    - Merge such terms in the segment tree.
    - Output the value of the whole expression.
- Total running time: $O((N + M) \log N)$

Scientific Grading

Problem

Given numbers x, y, a, b, c, d in the form $m \times 10^n$, determine if the relative and absolute errors of a, b, c, d for $x + y, x - y, xy, x/y$ (respectively) are less than $\varepsilon = 10^{-9}$.

Solution

- Observations:
  - A significand has at most 10 digits.
  - An exponent has the range of $[-10^9, 10^9]$.
    - We cannot use built-in data types directly.
  - Division can be hard to implement.

Scientific Grading (cont.)

Solution

- We can test numbers without any divisions.
  - Addition: $|x + y - a| < \varepsilon \cdot \min\{1, |x + y|\}$ or $x + y = a = 0$
  - Deletion: $|x - y - b| < \varepsilon \cdot \min\{1, |x - y|\}$ or $x - y = b = 0$
  - Multiplication: $|xy - c| < \varepsilon \cdot \min\{1, |xy|\}$
  - Division: $|x - yd| < \varepsilon \cdot \min\{|x|, |y|\}$
- Data structure: "big" decimal with negative digits
  - Represent a number as the sum of $\{a_i \cdot 10^i\}$ with $-9 \leq a_i \leq 9$.
  - Implement *addition*, *multiplication*, and *less-than*.
  - Example:
    - $1020 = 1 \cdot 10^3 + 2 \cdot 10^1$
    - $25 = 2 \cdot 10^1 + 5 \cdot 10^0$
    - $1020 - 25 = 1 \cdot 10^3 + (-5) \cdot 10^0$
- Other approaches available with careful case analysis

Building Roads

Problem

Given N points on a plane, find the diameter of the minimum diameter spanning tree.

Solution

- Assuming $N > 2$, one can prove that there is always a minimum diameter spanning tree that has no more than 3 edges in the diameter.
- That means this minimum diameter spanning tree is either:
  - a "star" with a centre connecting to every other vertex
  - two stars connected to each other.

Building Roads (cont.)

Solution

- For a star:
  - try every vertex as a centre
  - the diameter is the path connecting the two furthest vertices
  - $O(N^2)$
- For two stars:
  - for each vertex v, compute a list L_v of all other vertices sorted by distance from that vertex.
  - try every pair (u, v) of vertices as the two centres
  - figure out which of the two centres each of the other vertices connect to.
  - this can be done in linear time by successively increasing the maximum distance of the vertices connected to u by scanning L_u and assuming that everything else connects to v.
  - $O(N^3)$